

A local bias approach to the clustering of discrete density peaks

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Maxima of the linear density field form a point process that can be used to understand the spatial distribution of virialized halos that collapsed from initially overdense regions. However, owing to the peak constraint, clustering statistics of discrete density peaks are difficult to evaluate. For this reason, local bias schemes have received considerably more attention in the literature thus far. In this paper, we show that the 2-point correlation function of maxima of a homogeneous and isotropic Gaussian random field can be thought of, up to second order at least, as arising from a local bias expansion formulated in terms of rotationally invariant variables. This expansion relies on a unique smoothing scale, which is the Lagrangian radius of dark matter halos. The great advantage of this local bias approach is that it circumvents the difficult computation of joint probability distributions. We demonstrate that the bias factors associated with these rotational invariants can be computed using a peak-background split argument, in which the background perturbation shifts the corresponding probability distribution functions. Consequently, the bias factors are orthogonal polynomials averaged over those spatial locations that satisfy the peak constraint. In particular, asphericity in the peak profile contributes to the clustering at quadratic and higher order, with bias factors given by generalized Laguerre polynomials. We speculate that our approach remains valid at all orders, and that it can be extended to describe clustering statistics of any point process of a Gaussian random field. Our results will be very useful to model the clustering of discrete tracers with more realistic collapse prescriptions involving the tidal shear for instance.

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I. INTRODUCTION

In the biasing scenario introduced by [1], virialized halos form out of initially overdense regions with a linear density (extrapolated to the redshift of interest) equal to $\delta_c \approx 1.686$. Since then, this picture has received considerable support from observational data. Even though dark matter halos are extended objects, they form a spatial point process as far as their clustering is concerned. However, this essential feature has remained elusive in most theoretical descriptions of halo clustering, which assume that halos are a Poisson sampling of a more fundamental, continuous halo density field $\delta_h(\mathbf{x})$.

The peak formalism first proposed by [2, 3] in a cosmological context is interesting because it is a well-behaved point process. In this approach, virialized halos are associated with maxima of the initial density field. The displacement from their initial (Lagrangian) to final (Eulerian) position can be computed upon assuming phase space conservation [4]. Clustering statistics of these discrete density peaks display many of the features present in measurements of halo clustering extracted from N-body simulations. In particular, discrete density peaks exhibit a k -dependent linear bias factor [5, 6], small-scale exclusion [7, 8], and a linear velocity bias [9] etc. Some of these predictions have recently been tested in numerical simulations [10, 11]: peaks of the linear density field appear to provide a good approximation to the formation

sites of dark matter halos with $M \gtrsim M_*$.

However, despite recent progress towards the computation of peak clustering statistics [4] and a formulation of peak theory within the excursion set formalism [12, 13], discrete density peaks lack a clear connection with the more conventional local bias schemes [14], in which halos are approximated as a continuous field. Furthermore, while in the local bias model the computation of halo correlation functions is straightforward (though there are ambiguities regarding the filtering scale etc.), in the peak formalism calculations are particularly tedious owing to the peak constraint [3, 4, 6, 15]. In the most comprehensive analysis thus far, ref. [4] succeeded in computing the peak 2-point correlation $\xi_{pk}(r)$ up to second order, including the Zel'dovich displacement. They showed that some of the first- and second-order contributions could be obtained from a peak-background split formulated in terms of conditional mass functions. In contrast to most analytic models of halo clustering, which assume that the (k -independent) bias coefficients are the peak-background split biases, they derived this equivalence from first principles. However, they could not determine the physical origin of the other second-order contributions. Moreover, the peak constraint is clearly too simplistic to describe the clustering of low mass halos. In this mass range, one should consider more elaborated constraints involving the tidal shear etc. In this regards, it would be very desirable to find a simpler way of computing the correlation functions of generic point processes of a (Gaussian) random field.

In this paper, we suggest a simple, physically motivated prescription based on the peak-background split

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to compute the correlation functions of generic point processes driven by homogeneous and isotropic Gaussian random fields. We argue that clustering statistics of such point processes can be reduced to the evaluation of correlators of an *effective* continuous overdensity which, in the case of discrete peaks, is a function of the local (smoothed) mass density field and its derivatives. Our approach combines in a single coherent picture peak theory, peak-background split, local bias and the excursion set framework. For sake of clarity, we will focus on the 2-point correlation function of initial density peaks as computed in [4] to explain the fundamentals of our approach.

The paper is organized as follows. Sec. II furnishes a brief summary of clustering in peak theory. Sec. III is the central Section of the paper, where we present the connection between rotational invariants, peak-background split and a local peak bias prescription. Finally, Sec. IV discusses the implications of our findings.

II. CORRELATION FUNCTIONS FOR DENSITY PEAKS

We begin with a short recapitulation of the computation of correlation functions in the peak formalism. Let δ_s be the linear mass density field smoothed on scale R_s with a spherically symmetric filter. For convenience, we work with the normalized variables $\nu(\mathbf{x}) \equiv \frac{1}{\sigma_0} \delta_s(\mathbf{x})$, $\eta_i(\mathbf{x}) \equiv \frac{1}{\sigma_1} \partial_i \delta_s(\mathbf{x})$ and $\zeta_{ij}(\mathbf{x}) \equiv \frac{1}{\sigma_2} \partial_i \partial_j \delta_s(\mathbf{x})$. Here, ν is the peak height or significance, and

$$\sigma_n^2(R_s) \equiv \frac{1}{2\pi^2} \int_0^\infty dk k^{2(n+1)} P_s(k) . \quad (1)$$

are moments of the power spectrum $P_s(k) = \langle |\delta_s(\mathbf{k})|^2 \rangle$. A Gaussian filter is frequently adopted to ensure convergence of all the spectral moments σ_n , but one should bear in mind that the peak height associated with dark matter halos is always computed with a tophat filter (see [13] for details). The first few spectral moments σ_n can be combined into a dimensionless spectral width $\gamma_1 = \sigma_1^2/(\sigma_0\sigma_1)$ that takes values between zero and unity. γ_1 reflects the range over which the smoothed power spectrum $P_s(k)$ is significant, i.e. $\gamma_1 \approx 1$ for a sharply peaked power spectrum whereas $\gamma_1 \approx 0$ for a power spectrum that covers a wide range of wavenumbers.

Correlations of density maxima of δ_s can be evaluated using the Kac-Rice formula [16, 17]. The trick is to Taylor-expand $\eta_i(\mathbf{x})$ around the position \mathbf{x}_{pk} of a local maximum. As a result, the number density of (BBKS) peaks of height ν' at position \mathbf{x} in the smoothed density field δ_s can be expressed in terms of the field δ_s and its derivatives:

$$n_{\text{pk}}(\nu', R_s, \mathbf{x}) \equiv \frac{3^{3/2}}{R_s^3} |\det \zeta(\mathbf{x})| \delta_D[\boldsymbol{\eta}(\mathbf{x})] \theta_H[\lambda_3(\mathbf{x})] \times \delta_D[\nu(\mathbf{x}) - \nu'] , \quad (2)$$

where $R_\star \equiv \sqrt{3}(\sigma_1/\sigma_2)$ is the characteristic radius of a peak (and *not* the interpeak distance). The three-dimensional Dirac distribution $\delta_D(\boldsymbol{\eta})$ ensures that all extrema are included. The factors of theta function $\theta_H(\lambda_3)$, where λ_3 is the lowest eigenvalue of the shear tensor ζ_{ij} , and the Dirac delta $\delta_D(\nu - \nu')$ further restricts the set to density maxima of the desired significance ν' .

The (disconnected) N -point correlations $\rho_{\text{pk}}^{(N)}$ (or joint intensities) of density maxima are defined as the ensemble averages of products of $n_{\text{pk}}(\nu, R_s; \mathbf{x})$,

$$\rho_{\text{pk}}^{(N)}(\nu, R_s, \mathbf{x}_1, \dots, \mathbf{x}_N) \equiv \left\langle n_{\text{pk}}(\nu, R_s, \mathbf{x}_1) \times \dots \times n_{\text{pk}}(\nu, R_s, \mathbf{x}_N) \right\rangle . \quad (3)$$

For the Gaussian initial conditions considered here, multivariate normal distribution are assumed to perform the ensemble average. In the particular case $N = 1$, $\langle n_{\text{pk}}(\nu, R_s, \mathbf{x}) \rangle = \bar{n}_{\text{pk}}(\nu, R_s)$ is the average, differential number density of peaks of height ν identified on the filtering scale R_s [3],

$$\begin{aligned} \bar{n}_{\text{pk}}(\nu, R_s) &= \frac{1}{(2\pi)^2 R_\star^3} e^{-\nu^2/2} G_0^{(1)}(\gamma_1, \gamma_1 \nu) \\ &= \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} \left(\frac{1}{V_\star} \right) G_0^{(1)}(\gamma_1, \gamma_1 \nu) . \end{aligned} \quad (4)$$

In the last equality, $V_\star = (2\pi)^{3/2} R_\star^3$ is the typical 3-dimensional extent of a density peak [12]. The functions $G_n^{(\alpha)}(\gamma_1, \gamma_1 \nu)$ are defined in Appendix B. In particular, the ratio $G_k^{(1)}/G_0^{(1)}$ is equal to the k th moment $\overline{u^k}$ of the peak curvature u . Similarly, the reduced 2-point correlation function for maxima of a given significance separated by a distance $r = |\mathbf{r}| = |\mathbf{x}_2 - \mathbf{x}_1|$ is

$$\xi_{\text{pk}}(\nu, R_s, r) = \frac{\rho_{\text{pk}}^{(2)}(\nu, R_s, r)}{\bar{n}_{\text{pk}}^2(\nu, R_s)} - 1 , \quad (5)$$

Notice that, in $\rho_{\text{pk}}^{(2)}$, we have ignored the shot-noise term $\bar{n}_{\text{pk}} \delta_D(\mathbf{x}_2 - \mathbf{x}_1)$ that arises from the self-pairs as it matters only at zero-lag (in the peak power spectrum however, this contributes a constant Poisson noise $1/\bar{n}_{\text{pk}}$ at all wavenumbers).

The calculation of Eq.(5) at second order in the mass correlation and its derivatives is quite tedious [3, 4, 15] because one must evaluate the joint probability distribution for the 10-dimensional vector of variables $\mathbf{y}_\alpha^\top = (\eta_i(\mathbf{x}_\alpha), \nu(\mathbf{x}_\alpha), \zeta_A(\mathbf{x}_\alpha))$ at two different spatial locations $\mathbf{x}_\alpha = \mathbf{x}_1$ and \mathbf{x}_2 , i.e. a total of 20 variables. Here, the components ζ_A , $A = 1, \dots, 6$ symbolize the independent entries $ij = 11, 22, 33, 12, 13, 23$ of ζ_{ij} . Fortunately, as was shown in [4], most of the terms nicely combine together, so that the final result can be recast into the compact expression

$$\begin{aligned} \xi_{\text{pk}}(\nu, R_s, r) = & (\tilde{b}_I^2 \xi_0^{(0)}) + \frac{1}{2} (\xi_0^{(0)} \tilde{b}_{II}^2 \xi_0^{(0)}) - \frac{3}{\sigma_1^2} (\xi_1^{(1/2)} \tilde{b}_{II} \xi_1^{(1/2)}) - \frac{5}{\sigma_2^2} (\xi_2^{(1)} \tilde{b}_{II} \xi_2^{(1)}) \left(1 + \frac{2}{5} \partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1 \nu) \Big|_{\alpha=1} \right) \\ & + \frac{5}{2\sigma_2^4} \left[(\xi_0^{(2)})^2 + \frac{10}{7} (\xi_2^{(2)})^2 + \frac{18}{7} (\xi_4^{(2)})^2 \right] \left(1 + \frac{2}{5} \partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1 \nu) \Big|_{\alpha=1} \right)^2 \\ & + \frac{3}{2\sigma_1^4} \left[(\xi_0^{(1)})^2 + 2(\xi_2^{(1)})^2 \right] + \frac{3}{\sigma_1^2 \sigma_2^2} \left[3(\xi_3^{(3/2)})^2 + 2(\xi_1^{(3/2)})^2 \right]. \end{aligned} \quad (6)$$

The functions $\xi_\ell^{(n)}(r)$ are quantities analogous to σ_n^2 but defined for a finite separation r ,

$$\xi_\ell^{(n)}(R_s, r) = \frac{1}{2\pi^2} \int_0^\infty dk k^{2(n+1)} P_s(k) j_\ell(kr), \quad (7)$$

where $j_\ell(x)$ are spherical Bessel functions. In the right-hand side of Eq.(6), all the correlations depend on the filtering scale and the separation. However, the first line contains terms involving the first and second order peak bias parameters \tilde{b}_I and \tilde{b}_{II} (to be defined shortly), the second line retains a ν -dependence through the function $1 + (2/5)\partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1 \nu)|_{\alpha=1}$ solely, whereas the last two terms in the right-hand side depend on the separation r (and R_s) only. Hence, unlike standard local bias expansions (Eulerian or Lagrangian), the peak 2-point correlation also exhibits quadratic terms linear in the second-order bias \tilde{b}_{II} . These terms involve derivatives of the linear mass correlation $\xi_0^{(0)}$ and, therefore, vanish at zero lag. Clearly, they arise because the peak correlation also depends on the statistical properties of η_i and ζ_{ij} .

Ref. [4] also showed that the Lagrangian peak bias factors $\tilde{b}_N(k_1, \dots, k_N)$ can be constructed upon averaging over the peak curvature products of b_ν and b_u , where

$$b_\nu(\nu, R_s) = \frac{1}{\sigma_0} \left(\frac{\nu - \gamma_1 u}{1 - \gamma_1^2} \right), \quad (8)$$

$$b_u(\nu, R_s) = \frac{1}{\sigma_2} \left(\frac{u - \gamma_1 \nu}{1 - \gamma_1^2} \right). \quad (9)$$

For peak of significance ν on the smooting scale R_s , the first order bias \tilde{b}_I is defined as the Fourier space multiplication (we omit the dependence on R_s and ν for short-hand convenience) [6]

$$\tilde{b}_I(k) = b_{10} + b_{01} k^2 \quad (10)$$

$$\text{where } b_{10} = \bar{b}_\nu \quad \text{and} \quad b_{01} = \bar{b}_u.$$

The overline designates the average over the peak curvature. b_{01} can be quite large for moderate peak heights. In the high peak limit $\nu \gg 1$ however, it is negligible so that $\tilde{b}_I(k)$ is nearly scale-independent (like in local bias models). Similarly, the Fourier space expression of the second order peak bias \tilde{b}_{II} is [4]

$$\tilde{b}_{II}(k_1, k_2) = b_{20} + b_{11}(k_1^2 + k_2^2) + b_{02} k_1^2 k_2^2, \quad (11)$$

where k_1 and k_2 are wavemodes and the k -independent coefficients b_{20} , b_{11} and b_{02} are

$$b_{20}(\nu, R_s) \equiv \bar{b}_\nu^2 - \frac{1}{\sigma_0^2 (1 - \gamma_1^2)} \quad (12)$$

$$= \frac{1}{\sigma_0^2} \left[\frac{\nu^2 - 2\gamma_1 \nu \bar{u} + \gamma_1^2 \bar{u}^2}{(1 - \gamma_1^2)^2} - \frac{1}{(1 - \gamma_1^2)} \right]$$

$$b_{11}(\nu, R_s) \equiv \bar{b}_\nu \bar{b}_u + \frac{\gamma_1^2}{\sigma_1^2 (1 - \gamma_1^2)} \quad (13)$$

$$= \frac{1}{\sigma_0 \sigma_2} \left[\frac{(1 + \gamma_1^2) \nu \bar{u} - \gamma_1 [\nu^2 + \bar{u}^2]}{(1 - \gamma_1^2)^2} + \frac{\gamma_1}{(1 - \gamma_1^2)} \right],$$

and

$$b_{02}(\nu, R_s) \equiv \bar{b}_u^2 - \frac{1}{\sigma_2^2 (1 - \gamma_1^2)} \quad (14)$$

$$= \frac{1}{\sigma_2^2} \left[\frac{\bar{u}^2 - 2\gamma_1 \nu \bar{u} + \gamma_1^2 \nu^2}{(1 - \gamma_1^2)^2} - \frac{1}{(1 - \gamma_1^2)} \right].$$

By definition, \tilde{b}_{II}^m acts on the functions $\xi_{\ell_1}^{(n_1)}(r)$ and $\xi_{\ell_2}^{(n_2)}(r)$ as follows:

$$\begin{aligned} (\xi_{\ell_1}^{(n_1)} \tilde{b}_{II}^m \xi_{\ell_2}^{(n_2)}) & \equiv \frac{1}{4\pi^4} \int_0^\infty dk_1 \int_0^\infty dk_2 k_1^{2(n_1+1)} k_2^{2(n_2+1)} \\ & \times \tilde{b}_{II}^m(k_1, k_2) P_s(k_1) P_s(k_2) j_{\ell_1}(k_1 r) j_{\ell_2}(k_2 r). \end{aligned} \quad (15)$$

As pointed out by [4], the piece $\tilde{b}_I^2 \xi_0^{(0)} + (1/2) \xi_0^{(0)} \tilde{b}_{II}^2 \xi_0^{(0)}$ can be thought of as arising from the continuous, deterministic, local bias relation

$$\begin{aligned} \delta_{\text{pk}}(\mathbf{x}) = & b_{10} \delta_s(\mathbf{x}) - b_{01} \nabla^2 \delta_s(\mathbf{x}) + \frac{1}{2} b_{20} \delta_s^2(\mathbf{x}) \\ & - b_{11} \delta_s(\mathbf{x}) \nabla^2 \delta_s(\mathbf{x}) + \frac{1}{2} b_{02} [\nabla^2 \delta_s(\mathbf{x})]^2, \end{aligned} \quad (16)$$

where the bias factors b_{ij} are peak-background split bias factors that follow from expanding the conditional peak number density in a series in the small background density perturbation δ_l . This expansion is local in the sense that, except for the filtering, it involves quantities evaluated at \mathbf{x} solely. However, an essential difference with

the widespread local bias model [14] is the fact that, when computing the ensemble average $\langle \delta_{\text{pk}}(\mathbf{x}_1) \delta_{\text{pk}}(\mathbf{x}_2) \rangle$, we must ignore all powers of zero-lag moments (such as, e.g., σ_0^4 in $\langle \delta_s^2(\mathbf{x}_1) \delta_s^2(\mathbf{x}_2) \rangle$) to recover $\xi_{\text{pk}}(r)$ since the latter does not exhibit such contributions (this ‘no zero-lag requirement’ also arises in the derivation of the ‘renormalized’ bias parameters of [18]). All the terms in Eq.(16) are of course invariant under rotations since $\delta_{\text{pk}}(\mathbf{x})$ transforms as a scalar under rotations. Clearly however, this series expansion is not the most generic Lagrangian expansion we may conceive of (see, e.g., [19] non nonlocal Lagrangian bias).

Notwithstanding these results, [4] did not succeed in finding a physical interpretation of the other second-order terms in the right-hand side of Eq.(6), even though it was pretty clear that they – at least partially – arise from coupling involving the components of the gradient η_i and the hessian ζ_{ij} .

III. A INTUITIVE INTERPRETATION OF $\xi_{\text{pk}}(r)$

In this Section, we propose an intuitive, physically motivated explanation of Eq. (6) that is grounded in the peak-background split argument [1]. We begin with a brief introduction to the helicity basis, which was used in [4] to compute probability distributions of the density field and its derivatives at two different spatial locations.

A. Probability density in the helicity basis

The 2-point correlation function of initial density peaks is the ensemble average of $n_{\text{pk}}(\mathbf{y}_1) n_{\text{pk}}(\mathbf{y}_2)$ over the joint probability density $P_2(\mathbf{y}_1, \mathbf{y}_2; r)$, where $\mathbf{y}_\alpha \equiv \mathbf{y}(\mathbf{x}_\alpha)$ are the values of the field and its derivatives at position \mathbf{x}_α . In what follows, $n_{\text{pk}}(\mathbf{y})$ will also designate Eq.(2). Following [4], we can decompose the variables $\mathbf{y}_\alpha^\top = (\eta_i(\mathbf{x}_\alpha), \nu(\mathbf{x}_\alpha), \zeta_A(\mathbf{x}_\alpha))$ that appear in the joint probability density $P_2(\mathbf{y}_1, \mathbf{y}_2; r)$ in the helicity basis $(\mathbf{e}_+, \hat{\mathbf{r}}, \mathbf{e}_-)$, where

$$\mathbf{e}_+ \equiv \frac{i\hat{\mathbf{e}}_\phi - \hat{\mathbf{e}}_\theta}{\sqrt{2}}, \quad \hat{\mathbf{r}} \equiv \mathbf{r}/r, \quad \mathbf{e}_- \equiv \frac{i\hat{\mathbf{e}}_\phi + \hat{\mathbf{e}}_\theta}{\sqrt{2}} \quad (17)$$

and $\hat{\mathbf{r}}$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\phi$ are orthonormal vectors in spherical coordinates (r, θ, ϕ) . The orthogonality relations between these vectors are $\mathbf{e}_\pm \cdot \mathbf{e}_\pm = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$ and $\mathbf{e}_+ \cdot \mathbf{e}_- = \mathbf{e}_\pm \cdot \hat{\mathbf{r}} = 0$, where the inner product between two vectors \mathbf{u} and \mathbf{v} is defined as $\mathbf{u} \cdot \mathbf{v} \equiv u_i \bar{v}_i \equiv u_i v^i$. Unless otherwise stated, an overline will denote complex conjugation throughout Sec. III A.

In this reference frame, we decompose the first derivatives as

$$\boldsymbol{\eta} \equiv \eta^{(0)} \hat{\mathbf{r}} + \eta^{(+1)} \hat{\mathbf{e}}_+ + \eta^{(-1)} \mathbf{e}_-. \quad (18)$$

Here, $\eta^{(0)} \equiv \boldsymbol{\eta} \cdot \hat{\mathbf{r}}$ and $\eta^{(\pm 1)} \equiv \boldsymbol{\eta} \cdot \mathbf{e}_\pm$ are the helicity-0 and -1 components. The correlation properties of $\eta^{(0)}$ and

$\eta^{(\pm 1)}$ can be obtained by projecting out the scalar and vector parts of the correlation of the Cartesian components η_i with the projection operator $P = \mathbf{e}_+ \otimes \bar{\mathbf{e}}_+ + \mathbf{e}_- \otimes \bar{\mathbf{e}}_-$. The rule of thumb is that $\langle \eta_1^{(s)} \bar{\eta}_2^{(s')} \rangle = \langle \eta_1^{(s)} \eta_2^{(-s')} \rangle$, where $\eta_\alpha^{(s)} = \eta^{(s)}(\mathbf{x}_\alpha)$, vanish unless $s - s' = 0$. We find

$$\begin{aligned} \langle \eta_1^{(0)} \bar{\eta}_2^{(0)} \rangle &= \frac{1}{3\sigma_1^2} (\xi_0^{(1)} - 2\xi_2^{(1)}) \\ \langle \eta_1^{(\pm 1)} \bar{\eta}_2^{(\pm 1)} \rangle &= \frac{1}{3\sigma_1^2} (\xi_0^{(1)} + \xi_2^{(1)}) \\ \langle \eta_1^{(\pm 1)} \bar{\eta}_2^{(\mp 1)} \rangle &= 0. \end{aligned} \quad (19)$$

Here and henceforth, the subscripts “1” and “2” will denote variables evaluated at position \mathbf{x}_1 and \mathbf{x}_2 for short-hand convenience. Similarly, the symmetric tensor ζ_{ij} can be decomposed into its trace and traceless components,

$$\begin{aligned} \zeta_{ij} &\equiv -\frac{1}{3} u \delta_{ij} + \tilde{\zeta}_{ij} \\ \tilde{\zeta}_{ij} &= S_{ij} \zeta^{(S)} + \sqrt{\frac{1}{3}} (\zeta_i^{(V)} \hat{r}_j + \zeta_j^{(V)} \hat{r}_i) + \sqrt{\frac{2}{3}} \zeta_{ij}^{(T)}. \end{aligned} \quad (20)$$

The variables $u \equiv -\text{tr} \zeta = -\zeta_i^i$ and $\zeta^{(S)} \equiv \zeta^{(0)}$ are the longitudinal and transverse helicity-0 modes, $\zeta_i^{(V)}$ are the components of a transverse vector, $\zeta^{(V)} \cdot \hat{\mathbf{r}} = 0$, whereas $\zeta_{ij}^{(T)}$ is a symmetric, traceless, transverse tensor, $\delta^{ij} \zeta_{ij}^{(T)} = \zeta_{ij}^{(T)} \hat{r}^j = 0$. Explicit expressions for these variables are

$$\zeta^{(S)} \equiv \frac{3}{2} S^{lm} \zeta_{lm} = \frac{1}{2} (3\hat{r}^l \hat{r}^m - \delta^{lm}) \zeta_{lm} \quad (21)$$

$$\zeta_i^{(V)} \equiv \sqrt{3} V_i^{lm} \zeta_{lm} = \sqrt{3} (\delta_i^l - \hat{r}_i \hat{r}^l) \hat{r}^m \zeta_{lm} \quad (22)$$

$$\begin{aligned} \zeta_{ij}^{(T)} &\equiv \sqrt{\frac{3}{2}} T_{ij}^{lm} \zeta_{lm} \\ &= \sqrt{\frac{3}{2}} \left(P_i^l P_j^m - \frac{1}{2} P_{ij} P^{lm} \right) \zeta_{lm}, \end{aligned} \quad (23)$$

where S_{ab} , V_a^{bc} and T_{ab}^{cd} are the scalar, vector and tensor projections operators (see, e.g., [20]). We have introduced factors of $\sqrt{1/3}$ and $\sqrt{2/3}$ in the decomposition Eq.(20) such that the zero-point moments of the helicity-0, -1 and -2 variables all equal 1/5 (see Eq. (24) below). The helicity-1 components of $\zeta^{(V)}$ and their complex conjugates are given by $\zeta^{(\pm 1)} \equiv \zeta^{(V)} \cdot \mathbf{e}_\pm = \sqrt{3} e_\pm^i \hat{r}^j \zeta_{ij}$ and $\bar{\zeta}^{(\pm 1)} \equiv \zeta^{(V)} \cdot \bar{\mathbf{e}}_\pm = \sqrt{3} \bar{e}_\pm^i \hat{r}^j \zeta_{ij}$, whereas $\zeta^{(\pm 2)} \equiv \zeta_{ij}^{(T)} e_\pm^i e_\pm^j = \sqrt{3/2} e_\pm^i e_\pm^j \zeta_{ij}$ and $\bar{\zeta}^{(\pm 2)} \equiv \zeta_{ij}^{(T)} \bar{e}_\pm^i \bar{e}_\pm^j = \sqrt{3/2} \bar{e}_\pm^i \bar{e}_\pm^j \zeta_{ij}$ are the two independent helicity-2 modes (polarizations) and their complex conjugates, respectively. Hereafter designating $\zeta^{(s)}(\mathbf{x}_\alpha)$ as $\zeta_\alpha^{(s)}$, the cor-

relation properties of these variables are the following:

$$\begin{aligned}\langle \zeta_1^{(0)} \zeta_2^{(0)} \rangle &= \frac{1}{\sigma_2^2} \left(\frac{1}{5} \xi_0^{(2)} - \frac{2}{7} \xi_2^{(2)} + \frac{18}{35} \xi_4^{(2)} \right) \\ \langle \zeta_1^{(\pm 1)} \bar{\zeta}_2^{(\pm 1)} \rangle &= \frac{1}{\sigma_2^2} \left(\frac{1}{5} \xi_0^{(2)} - \frac{1}{7} \xi_2^{(2)} - \frac{12}{35} \xi_4^{(2)} \right) \\ \langle \zeta_1^{(\pm 2)} \bar{\zeta}_2^{(\pm 2)} \rangle &= \frac{1}{\sigma_2^2} \left(\frac{1}{5} \xi_0^{(2)} + \frac{2}{7} \xi_2^{(2)} + \frac{3}{35} \xi_4^{(2)} \right),\end{aligned}\quad (24)$$

and $\langle \zeta_1^{(s)} \zeta_2^{(s')} \rangle = \langle \zeta_1^{(s)} \bar{\zeta}_2^{(-s')} \rangle$. All the other correlations vanish. Note that the covariances are real despite the fact that the helicity-1 and -2 variables are complex.

While the average peak number density only depends on the matrix M of covariances at the same location, the computation of the peak 2-point correlation function $\xi_{\text{pk}}(r)$ and higher-order clustering statistics from Eq. (3) generally involve covariances of the random fields at different locations. For $\xi_{\text{pk}}(r)$, the covariance matrix $C(r) \equiv \langle \mathbf{y} \mathbf{y}^\dagger \rangle$, where $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$, is a 20-dimensional matrix that may be partitioned into four 10×10 block matrices: the zero-point contribution M in the top left and bottom right corners, and the cross-correlation matrix $B(r)$ and its transpose in the bottom left and top right corners, respectively. Expressions for M and $B(r)$ in the helicity basis can be found in the Appendix of [4].

B. Rotational invariants

Translational and rotational invariance implies that \bar{n}_{pk} does not depend on spatial position, and that $\xi_{\text{pk}}(r)$ be a function of the distance r solely. In this regards, [4] noted that, although the covariance matrix $C(r)$ in the helicity basis (17) does not depend on the direction $\hat{\mathbf{r}}$ of the separation vector \mathbf{r} , it is not equal to the angular average covariance matrix $\hat{C}(r) \equiv (1/4\pi) \int d\Omega_{\hat{\mathbf{r}}} C(\mathbf{r})$. The

latter is obtained upon setting $\xi_\ell^{(n)} \equiv 0$ whenever $\ell \neq 0$ in the expression of $B(r)$. As a consequence, $\hat{C}(r)$ retains the correlations $\langle \nu_1 \nu_2 \rangle$, $\langle \nu_1 u_2 \rangle$, $\langle u_1 u_2 \rangle$ and parts of the covariances $\langle \eta_1^{(m)} \bar{\eta}_2^{(m)} \rangle$ and $\langle \zeta_1^{(m)} \bar{\zeta}_2^{(m)} \rangle$. However, [4] did not provide a convincing explanation for their observation. We shall do it now.

Firstly, it is pretty clear that, since the peak 2-point correlation is invariant under rotations of the reference frame, it should be possible to express it in terms of rotational invariants constructed from the variables ν , η_i and ζ_{ij} . The peak significance ν and the trace u are two obvious candidates, but they are not the only ones. The vector $\boldsymbol{\eta}$ of first derivatives and the traceless matrix $\tilde{\zeta}$ yield two additional invariants, i.e. the square modulus $\boldsymbol{\eta} \cdot \boldsymbol{\eta}$ and the trace $\text{tr}(\tilde{\zeta}^2)$. In the helicity basis, these invariants can be written

$$\boldsymbol{\eta} \cdot \boldsymbol{\eta} = \eta^{(0)} \eta^{(0)} + \eta^{(+1)} \bar{\eta}^{(+1)} + \eta^{(-1)} \bar{\eta}^{(-1)} \quad (25)$$

and

$$\begin{aligned}\text{tr}(\tilde{\zeta}^2) &= \frac{2}{3} \left[\zeta^{(0)} \zeta^{(0)} \right. \\ &\quad \left. + \sum_{s=1,2} \left(\zeta^{(+s)} \bar{\zeta}^{(+s)} + \zeta^{(-s)} \bar{\zeta}^{(-s)} \right) \right].\end{aligned}\quad (26)$$

The 3×3 symmetric matrix ζ_{ij} actually provides a third invariant with respect to rotations: the determinant $\det \zeta$. However, as we shall see below in the discussion of the peak-background split, because this determinant only enters the peak number density $n_{\text{pk}}(\mathbf{y})$ and not the 1-point multivariate normal distribution $P_1(\mathbf{y})$ (the argument of the exponential is quadratic in the variables), it does not contribute directly to the peak bias. This suggests that we look at the covariances of $\eta^2(\mathbf{x})$ and $\zeta^2(\mathbf{x})$, where these are defined as

$$\eta^2(\mathbf{x}) \equiv \boldsymbol{\eta}(\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{x}) = -\frac{1}{\sigma_1^2} \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \delta_{ij} k_{1i} k_{2j} \delta_s(\mathbf{k}_1) \delta_s(\mathbf{k}_2) e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}}, \quad (27)$$

$$\zeta^2(\mathbf{x}) \equiv \frac{3}{2} \text{tr}[\tilde{\zeta}^2(\mathbf{x})] = \frac{3}{2\sigma_2^2} \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \left(\delta_{il} \delta_{jm} - \frac{1}{3} \delta_{ij} \delta_{lm} \right) k_{1i} k_{1j} k_{2l} k_{2m} \delta_s(\mathbf{k}_1) \delta_s(\mathbf{k}_2) e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}}, \quad (28)$$

where $\delta_s(\mathbf{k})$ are the Fourier modes of the smoothed density field. As we shall see in Sec. III C, the variables $3\eta^2$ and $5\zeta^2$ are distributed as chi-squared (χ^2) variates with 3 and 5 degrees of freedom, respectively. Using either the Fourier space expression of $\eta_\alpha^2 \equiv \eta^2(\mathbf{x}_\alpha)$ (not to be confounded here with a cartesian component of the vector $\boldsymbol{\eta}$) or the fact that only components of identical helicity correlate, it is straightforward to compute the following correlators (for illustrative purposes, Appendix

A furnishes details of the calculation of $\langle \eta_1^2 \eta_2^2 \rangle$):

$$\langle \eta_1^2 \eta_2^2 \rangle = 1 + \frac{2}{3\sigma_1^4} \left[(\xi_0^{(1)})^2 + 2(\xi_2^{(1)})^2 \right] \quad (29)$$

$$\langle \eta_1^2 \nu_2^2 \rangle = 1 + \frac{2}{\sigma_0^2 \sigma_1^2} \xi_1^{(1/2)} \xi_1^{(1/2)}. \quad (30)$$

Ignoring the zero-lag contributions, these terms correspond exactly (up to a sign factor) to some of those entering the second-order contribution of $\xi_{\text{pk}}(r)$ in Eq.(6),

with $(\xi_1^{(1/2)} b_{20} \xi_1^{(1/2)})$ being proportional to $\langle \eta_1^2 \nu_2^2 \rangle$ in particular. The computations of correlators involving $\zeta_\alpha^2 \equiv \zeta^2(\mathbf{x}_\alpha)$ proceeds in a similar way although, in this case, it is much easier to sum the correlations among equal helicity components. For instance,

$$\langle \zeta_1^2 \zeta_2^2 \rangle = 1 + 2 \left\{ \langle \zeta_1^{(0)} \zeta_2^{(0)} \rangle^2 + 2 \sum_{s=1,2} \left(\langle \zeta_1^{(+s)} \zeta_2^{(+s)} \rangle^2 + \langle \zeta_1^{(-s)} \zeta_2^{(-s)} \rangle^2 \right) \right\}, \quad (31)$$

where we used the fact that $\langle \zeta_1^{(s)} \zeta_2^{(-s)} \rangle = \langle \zeta_1^{(s)} \zeta_2^{(s)} \rangle$. After some algebra, we find

$$\langle \zeta_1^2 \zeta_2^2 \rangle = 1 + \frac{2}{5\sigma_2^4} \left[(\xi_0^{(2)})^2 + \frac{10}{7} (\xi_2^{(2)})^2 + \frac{18}{7} (\xi_4^{(2)})^2 \right], \quad (32)$$

and

$$\langle \zeta_1^2 \eta_2^2 \rangle = 1 + \frac{2}{5\sigma_1^2 \sigma_2^2} \left[3(\xi_3^{(3/2)})^2 + 2(\xi_1^{(3/2)})^2 \right] \quad (33)$$

$$\langle \zeta_1^2 \nu_2^2 \rangle = 1 + \frac{2}{\sigma_0^2 \sigma_2^2} \xi_2^{(1)} \xi_2^{(1)}. \quad (34)$$

The cross-correlations of η_1^2 and ζ_1^2 with u_2 in place of ν_2 are identical except for the superscript (n) , which should be replaced by $(n+1)$. Again, all these terms can be found among the second-order contributions in the right-hand side of Eq. (6).

Therefore, the actual dependence of $\xi_{\text{pk}}(r)$ on the invariants $\eta^2(\mathbf{x})$ and $\zeta^2(\mathbf{x})$, whose covariances involve the correlation functions $\xi_\ell^{(n)}$ with $\ell \neq 0$, is the fundamental reason for $C(r)$ being different from the angle average $\bar{C}(r)$. Those correlations, which arise upon expanding the joint probability density at second order, eventually all nicely combine together (and with terms proportional to $(\xi_0^{(1)})^2$ and $(\xi_0^{(2)})^2$) to yield the second-order correlators $\langle \eta_1^2 \nu_2^2 \rangle$, $\langle \zeta_1^2 \eta_2^2 \rangle$ etc.

The question then arises of the calculation of the coefficients of these quadratic terms in the peak 2-point correlation function. We already know that the coefficients multiplying products of the form $\xi_0^{(n)} \xi_0^{(n')}$ are the quadratic peak-background split biases associated to the scalars ν and u . Does this hold also for the coefficients multiplying $\langle \eta_1^2 \eta_2^2 \rangle$, $\langle \zeta_1^2 \nu_2^2 \rangle$ etc. ?

C. Generalizing the peak background-split

The probability distribution $P_1(\mathbf{y})$ that is needed to compute \bar{n}_{pk} is a multivariate Gaussian of covariance matrix \mathbf{M} ,

$$P_1(\mathbf{y}) d^{10} \mathbf{y} = \frac{1}{(2\pi)^5 |\det \mathbf{M}|^{1/2}} e^{-Q_1(\mathbf{y})} d^{10} \mathbf{y}. \quad (35)$$

Owing to rotational invariance, this probability density is a function of ν , u , η^2 and ζ^2 solely (see, e.g., [21] for a systematic analysis of distribution functions of homogeneous and isotropic random fields). The quadratic form $Q_1(\mathbf{y})$ that appears in the exponential factor reads

$$Q_1(\mathbf{y}) = \frac{\nu^2 + u^2 - 2\gamma_1 \nu u}{2(1 - \gamma_1^2)} + \frac{3}{2} \eta^2 + \frac{5}{2} \zeta^2, \quad (36)$$

so that $\exp[-Q_1(\mathbf{y})]$ retains factorization with respect to (ν, u) , η^2 and ζ^2 . Furthermore, since $\eta^{(s)}$ (with $s = \pm 1$) and $\zeta^{(s)}$ (with $s = \pm 1, \pm 2$) are complex random variables with mean 0 and variance 1/3 and 1/5 respectively and since $\langle \nu \zeta^{(0)} \rangle = \langle \nu \eta^{(0)} \rangle = 0$ and $\langle \eta^{(0)} \zeta^{(0)} \rangle = 0$ at the same spatial location, the quantities $3\eta^2(\mathbf{x})$ and $5\zeta^2(\mathbf{x})$ are independent χ^2 -distributed variables with 3 and 5 degrees of freedom, respectively (similar conclusions can be drawn for the distribution of the components of the deformation tensor, see [19, 22]). Therefore, the 1-point probability density can also be written as

$$P_1(\mathbf{y}) d^{10} \mathbf{y} = \mathcal{N}(\nu, u) d\nu du \times \chi_3^2(3\eta^2) d(3\eta^2) \chi_5^2(5\zeta^2) d(5\zeta^2), \quad (37)$$

where $\mathcal{N}(\nu, u)$ is the bivariate normal

$$\mathcal{N}(\nu, u) = \frac{1}{2\pi \sqrt{1 - \gamma_1^2}} \exp \left[-\frac{\nu^2 + u^2 - 2\gamma_1 \nu u}{2(1 - \gamma_1^2)} \right], \quad (38)$$

and

$$\chi_k^2(x) = \frac{1}{2^{k/2} \Gamma(k/2)} x^{k/2-1} e^{-x/2} \quad (39)$$

is a χ^2 -distribution with k degrees of freedom. Note that the distribution of ζ^2 is coupled with the last rotational invariant $\det \zeta$ [23]. However, it can be easily checked that, upon integrating over the (uniform) distribution of $\det \zeta$, we obtain the χ^2 -distribution $\chi_5^2(5\zeta^2)$.

Ref. [4] discussed how the peak bias factors b_{ij} can be derived from a peak-background split. They argued that, while the k -dependent piece b_{n0} is related to the n th order derivative of the differential number density \bar{n}_{pk} , derivatives cannot produce the bias factors b_{01} , b_{11} etc. multiplying the k -dependent terms. For this reason, they considered a second implementation of the peak-background split [24, 25] in which the dependence of the mass function on the overdensity of the background is derived explicitly. However, it is possible to write all the peak bias factors b_{ij} as derivatives of $\mathcal{N}(\nu, u)$ rather than \bar{n}_{pk} . More precisely, the b_{ij} are the bivariate Hermite polynomials

$$H_{ij}(\nu, u) = \mathcal{N}(\nu, u)^{-1} \left(-\frac{\partial}{\partial \nu} \right)^i \left(-\frac{\partial}{\partial u} \right)^j \mathcal{N}(\nu, u) \quad (40)$$

relative to the weight $\mathcal{N}(\nu, u)$, further averaged over the peak curvature u . Therefore, they are peak-background split biases in the sense that they can be derived from

the transformation $\nu \rightarrow \nu + \epsilon_1$ and $u \rightarrow u + \epsilon_2$, where ϵ_i and ϵ_2 are long-wavelength background perturbations uncorrelated with the (small-scale) fields $\nu(\mathbf{x})$ and $u(\mathbf{x})$. Ref. [4] did not express the peak-background split this way because they considered the effect of a background perturbation *after* the integration over the peak curvature. In terms of the rotational invariants introduced above, we can write the b_{ij} as

$$\sigma_0^i \sigma_2^j b_{ij} = \frac{1}{\bar{n}_{\text{pk}}} \int d^{10}\mathbf{y} n_{\text{pk}}(\mathbf{y}) H_{ij}(\nu, u) P_1(\mathbf{y}), \quad (41)$$

where it is understood that $P_1(\mathbf{y})$ takes the form Eq. (37) and $d^{10}\mathbf{y} = d\nu du d(3\eta^2) d(5\zeta^2)$. Factors of $1/\sigma_0$ and $1/\sigma_2$ are introduced because bias factors are ordinarily defined relative to the physical field $\delta_s(\mathbf{x})$ and its derivatives rather than the normalized variables. In practice, the integral is most easily performed upon transforming the 5 degrees of freedom attached to $5\zeta^2$ to the shape parameters v and w and the 3 Euler angles that describe the orientation of the principal axis frame (see Appendix B for details).

In [23], it was noticed that, in the presence of non-Gaussianity, the 1-point probability density $P_1(\mathbf{y})$ can be expanded in the set of orthogonal polynomials associated to the weight provided by $P_1(\mathbf{y})$ in the Gaussian limit. The same logic applies to the peak bias factors. Namely, the b_{ij} are drawn from the orthogonal polynomials associated to $\mathcal{N}(\nu, u)$, i.e. bivariate Hermite polynomials. Therefore, we expect that $\eta^2(\mathbf{x})$ and $\zeta^2(\mathbf{x})$ also generate bias parameters, and that these are drawn from the orthogonal polynomials associated with χ^2 -distributions, i.e. generalized Laguerre polynomials. These are defined as

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) \quad (42)$$

and are orthogonal over $[0, \infty[$ with respect to the χ^2 -distribution with $k = 2(\alpha + 1)$ degrees of freedom. The orthogonality relation can be expressed as

$$\int_0^\infty dx x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{mn}. \quad (43)$$

The first generalized Laguerre polynomials are $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = -x + \alpha + 1$.

Given the correlator Eq. (29), the term proportional to $1/\sigma_1^4$ in the right-hand side of Eq.(6) indicates that the first-order bias parameter χ_{10} associated with the invariant $\sigma_1^2 \eta^2(\mathbf{x}) = (\nabla \delta_s)^2(\mathbf{x})$ is $\chi_{10} = -3/(2\sigma_1^2)$. The aforementioned considerations suggest that we define the k th-order bias factor as the Laguerre polynomial $(-1)^k L_k^{(1/2)}(3\eta^2/2)$ averaged over all the possible peak configurations, i.e.

$$\sigma_1^{2k} \chi_{k0} \equiv \frac{(-1)^k}{\bar{n}_{\text{pk}}} \int d^{10}\mathbf{y} n_{\text{pk}}(\mathbf{y}) L_k^{(1/2)}\left(\frac{3\eta^2}{2}\right) P_1(\mathbf{y}). \quad (44)$$

Taking into account the peak constraint, the first-order bias factors thus is

$$\chi_{10} = \frac{1}{\sigma_1^2 \bar{n}_{\text{pk}}} \int d^{10}\mathbf{y} n_{\text{pk}}(\mathbf{y}) \left(\frac{3}{2}\eta^2 - \frac{3}{2}\right) P_1(\mathbf{y}) \quad (45)$$

$$= -\frac{3}{2\sigma_1^2}, \quad (46)$$

which is precisely what we were expecting. Similarly, we define the bias parameter χ_{0k} associated to the invariant $\zeta^2(\mathbf{x})$ as the ensemble average of the Laguerre polynomial $L_k^{(3/2)}(5\zeta^2/2)$ orthogonal with respect to the weight $\chi_5^2(5\zeta^2)$. Namely,

$$\sigma_2^{2k} \chi_{0k} \equiv \frac{(-1)^k}{\bar{n}_{\text{pk}}} \int d^{10}\mathbf{y} n_{\text{pk}}(\mathbf{y}) L_k^{(3/2)}\left(\frac{5\zeta^2}{2}\right) P_1(\mathbf{y}) \quad (47)$$

Note that, although we use a single symbol χ_{ij} to designate the bias factors derived from the χ^2 -distributions, the variables η^2 and ζ^2 , unlike ν and u , are uncorrelated. The first-order bias factor thus is

$$\chi_{01} = \frac{1}{\sigma_2^2 \bar{n}_{\text{pk}}} \int d^{10}\mathbf{y} n_{\text{pk}}(\mathbf{y}) \left(\frac{5}{2}\zeta^2 - \frac{5}{2}\right) P_1(\mathbf{y}). \quad (48)$$

To evaluate the integral, we first express the measure $d(5\zeta^2)$ in terms of the ellipticity v and prolateness w , so that ζ^2 can be written as $\zeta^2 = 3v^2 + w^2$ (see Appendix B for details). A multiplicative factor of ζ^2 will arise upon, e.g., taking the derivative of $\exp(-5\alpha\zeta^2/2)$ with respect to α . In the notation of [4], our factor of ζ^2 precisely corresponds to their derivative term $-(2/5)\partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1\nu)$ evaluated at $\alpha = 1$ (see Appendix B). Taking into account the factor of $G_0^{(1)}(\gamma_1, \gamma_1\nu)$ in the denominator, χ_{01} can eventually be written

$$\chi_{01} = -\frac{5}{2\sigma_2^2} \left(1 + \frac{2}{5} \partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1\nu) \Big|_{\alpha=1}\right). \quad (49)$$

The physical interpretation of this result is straightforward: $\zeta^2(\mathbf{x})$ is a scalar that describes the asymmetry of the peak density profile. In the high peak limit, $\chi_{01} \rightarrow 0$ reflecting the fact that the most prominent peaks are nearly spherical (see Fig.9 of [4]).

The physical origin for the appearance of these orthogonal polynomials can be found in the peak-background split. Long-wavelength background perturbations locally modulate the mean of the distributions $\mathcal{N}(\nu, u)$, $\chi_3^2(3\eta^2)$ and $\chi_5^2(5\zeta^2)$. The resulting non-central distributions can then be expanded in the appropriate set of orthogonal polynomials. In practice, it is convenient to introduce a shift or translation operator \hat{T}_ϵ to describe the action of a background perturbation on the distribution of rotational invariants. For the scalars ν and u , we define the shift operator as

$$\hat{T}_\epsilon \equiv \exp(-\epsilon_1 \partial_\nu - \epsilon_2 \partial_u), \quad (50)$$

where ϵ_1 and ϵ_2 are small perturbations to the peak significance and the peak curvature, i.e. $\nu \rightarrow \nu + \epsilon_1$ and

$u \rightarrow u + \epsilon_2$. The action of \hat{T}_ϵ on the probability density $\mathcal{N}(\nu, u)$ is to shift the (zero) mean of ν and u by $-\epsilon_1$ and $-\epsilon_2$, respectively (the reason for the minus sign is that Hermite polynomials include a factor of $(-1)^{i+j}$). A straightforward calculation gives

$$\begin{aligned} \mathcal{N}(\nu, u)^{-1} \hat{T}_\epsilon \mathcal{N}(\nu, u) &= \frac{\mathcal{N}(\nu - \epsilon_1, u - \epsilon_2)}{\mathcal{N}(\nu, u)} \\ &= f(\epsilon_1, \epsilon_2) e^{\epsilon_1 \sigma_0 b_\nu + \epsilon_2 \sigma_2 b_u}, \end{aligned} \quad (51)$$

where b_ν and b_u are given in Eqs.(8) and (9), and $f(\epsilon_1, \epsilon_2)$ is the exponential factor in $\mathcal{N}(\nu, u)$ with the replacement $\nu \rightarrow \epsilon_1$ and $u \rightarrow \epsilon_2$. The last expression is a generating function of bivariate Hermite polynomials. On expanding it in the small parameters ϵ_1 and ϵ_2 ,

$$\begin{aligned} \left\langle f(\epsilon_1, \epsilon_2) e^{\epsilon_1 \sigma_0 b_\nu + \epsilon_2 \sigma_2 b_u} \middle| \text{pk} \right\rangle \\ = \sum_{i,j=0}^{\infty} \sigma_0^i \sigma_2^j b_{ij} \left(\frac{\epsilon_1^i}{i!} \right) \left(\frac{\epsilon_2^j}{j!} \right), \end{aligned} \quad (52)$$

we recover the bias factors b_{ij} once the results are averaged over all locations that satisfy the peak constraint. Note that the bias parameter $b_{\nabla^2 \delta}$ defined in [18] bears the same physical meaning as our b_{01} : both represent the leading-order response of the tracer abundance to a uniform shift in the curvature of the density field.

For the quadratic variables η^2 and ζ^2 , Eq.(42) suggests that we express the shift operator in terms of both x and ∂_x . The definition is somewhat cumbersome because we must take into account not only the ordering of x and ∂_x , but also the factor of $\Gamma(n + \alpha + 1)/n!$ in the orthogonality relation Eq.(43). A sensible definition of \hat{T}_ϵ for the variable $x = 3\eta^2$ and $5\zeta^2$ is

$$\hat{T}_\epsilon \equiv : \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}k)}{j! \Gamma(\frac{1}{2}k + j)} \left(-\frac{\epsilon}{2} \partial_x x \right)^j : \quad (53)$$

$$= \Gamma(\frac{1}{2}k) : \frac{I_\alpha(\sqrt{-\frac{\epsilon}{2}} \partial_x x)}{(-\frac{\epsilon}{2} \partial_x x)^{\alpha/2}} : , \quad (54)$$

where $I_\alpha(x)$ is a modified Bessel function of the first kind and the symbol $::$ of normal ordering is borrowed from quantum field theory. In the present discussion, the normal ordering is defined as

$$: (\partial_x x)^n : f(x) \equiv \partial_x^n (x^n f(x)), \quad (55)$$

where $f(x)$ is some test function. With this definition, the action of \hat{T}_ϵ on a χ^2 -distribution with $k = 2(\alpha + 1)$

degrees of freedom is

$$\begin{aligned} \hat{T}_\epsilon \chi_k^2(x) &= \frac{1}{2} : \left[\sum_{j=0}^{\infty} \frac{(-\frac{\epsilon}{2})^j}{j! \Gamma(\frac{1}{2}k + j)} (\partial_x x)^j \right] : \left(\frac{x}{2} \right)^\alpha e^{-x/2} \\ &= \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-\frac{\epsilon}{2})^j}{j! \Gamma(\frac{1}{2}k + j)} \partial_x^j \left[\left(\frac{x}{2} \right)^{\alpha+j} e^{-x/2} \right] \\ &= \frac{e^{-x/2}}{2} \left(\frac{x}{2} \right)^\alpha \sum_{j=0}^{\infty} \frac{(-\frac{\epsilon}{2})^j}{\Gamma(\frac{1}{2}k + j)} L_j^{(\alpha)} \left(\frac{x}{2} \right) \\ &= \chi_k^2(x) \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}k)}{\Gamma(\frac{1}{2}k + j)} \left(-\frac{\epsilon}{2} \right)^j L_j^{(\alpha)} \left(\frac{x}{2} \right). \end{aligned} \quad (56)$$

This is precisely the Laguerre series expansion of a non-central χ^2 -variate derived in [26] (see Appendix C). Therefore,

$$[\chi_k^2(x)]^{-1} \hat{T}_\epsilon \chi_k^2(x) = \frac{\chi_k'^2(x; \epsilon)}{\chi_k^2(x)},$$

where $\chi_k'^2(x; \lambda)$ is a non-central χ^2 -distribution with k degrees of freedom and non-centrality parameter $\lambda \equiv \epsilon$. The latter is defined as the sum of squares $\lambda = \sum_{i=1}^k \mu_i^2$, where μ_i are the means of the random variables. We can now read off the bias factors from the expansion of $[\chi_k^2]^{-1} \hat{T}_\epsilon \chi_k^2$ in generalized Laguerre polynomials. For instance,

$$\left\langle \frac{\chi_5'^2(5\zeta^2; \epsilon_5)}{\chi_5^2(5\zeta^2)} \middle| \text{pk} \right\rangle = \sum_{j=0}^{\infty} \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2} + j)} \left(\frac{\epsilon_5}{2} \right)^j \sigma_2^{2j} \chi_{0j}. \quad (57)$$

Note that the more common generating function

$$(1 - \epsilon)^{-\alpha-1} \exp \left[\frac{x\epsilon}{2(1 - \epsilon)} \right] = \sum_{n=0}^{\infty} \epsilon^n L_n^{(\alpha)}(x) \quad (58)$$

appears to bear little connection with the non-central χ^2 -distribution.

To better understand the reason why the peak-background split generates a non-central χ^2 -distribution, we note that, owing to the relation $H_{2k}(x) \sim L_k(x^2)$ between Hermite and Laguerre polynomials, we could also have defined χ_{10} and χ_{01} as second derivatives of $P_1(\mathbf{y})$, where \mathbf{y} is now the vector $(\nu, u, \eta_i, \tilde{\zeta}_A)$ of independent normal random variables such that $\eta^2 = \sum_{i=1}^3 \eta_i^2$ and $\zeta^2 \equiv \sum_{A=1}^5 \tilde{\zeta}_A^2$. A little algebra shows that

$$\chi_{10} \equiv \frac{1}{\bar{n}_{\text{pk}}} \int d^{10} \mathbf{y} n_{\text{pk}}(\mathbf{y}) \sum_{j=1}^3 \left(\frac{1}{\sigma_1} \frac{\partial}{\partial \eta_j} \right)^2 P_1(\mathbf{y}), \quad (59)$$

Another way of writing this formula would be to absorb a factor of $1/\sqrt{2}$ in the definition of $\eta^{(\pm 1)}$ (and thus explicitly deal with complex normal distributions). Analogously, we have

$$\chi_{01} \equiv \frac{1}{\bar{n}_{\text{pk}}} \int d^{10} \mathbf{y} n_{\text{pk}}(\mathbf{y}) \sum_{A=1}^5 \left(\frac{1}{\sigma_2} \frac{\partial}{\partial \tilde{\zeta}_A} \right)^2 P_1(\mathbf{y}). \quad (60)$$

This suggests that the effect of a background perturbation on η^2 and ζ^2 can also be thought of as shifting the components of the first derivatives according to $\eta_i \rightarrow \eta_i + \epsilon_{3i}$, and those of $\tilde{\zeta}$ according to $\tilde{\zeta}_A \rightarrow \tilde{\zeta}_A + \epsilon_{5A}$. The small perturbations ϵ_{3i} and ϵ_{5A} need not be the same for distinct i and/or A . However, owing to invariance under rotations, only the length of the vector $\sum_{i=1}^3 (\epsilon_{3i})^2 \equiv \epsilon_3$ and $\sum_{A=1}^5 (\epsilon_{5A})^2 \equiv \epsilon_5$ matter. This is the reason why the background perturbation effectively shifts the respective χ^2 -distributions to non-central χ^2 -distributions, with non-centrality parameter $\lambda = \epsilon_3$ and $\lambda = \epsilon_5$. Note that it should be possible to formulate this peak-background split with the conditional peak number density $\bar{n}_{\text{pk}}(\nu, R_s | \delta_l, R_l)$ in a large-scale region of overdensity δ_l , like in [4]. However, one should then consider two long-wavelength perturbations η_l^2 and ζ_l^2 in order to describe the effect of the background perturbation on $\eta^2(\mathbf{x})$ and $\zeta^2(\mathbf{x})$, in addition to δ_l (which suffices to describe the effect of the background wave for both ν and u since these variables are correlated).

To conclude, [27] also pointed out that, even though there is no functional relation $n_{\text{pk}} = \mathcal{F}(\delta)$ for discrete density peaks, it is nevertheless possible to define renormalized bias parameters as the expectation values $c_n \sim \langle \mathcal{F} \rangle$. However, he did not compute them explicitly, nor specified what is \mathcal{F} (though it is pretty clear that it is related to n_{pk}). Here, we demonstrated explic-

itly that each of the combinations (ν, u) , η^2 and ζ^2 of rotational invariants generates a set of orthogonal polynomials which, upon taking the ensemble average over all the possible peak configurations, yields a set of bias factors. Furthermore, we showed that these bias parameters can be constructed from a suitable application of the peak-background split to the probability densities characterizing the invariants. We will now demonstrate that we can interpret the peak 2-point correlation as arising from a functional relation of the form $\delta_{\text{pk}} = \mathcal{F}(\delta_s, \dots)$.

D. A local bias approach to $\xi_{\text{pk}}(r)$

First, let us make sure that [4] obtained the correct expression for $\xi_{\text{pk}}(r)$. Adding all the second-order contributions induced by η^2 , ζ^2 and their cross-correlations with ν and u , our result differs from theirs in that the last term in the right-hand side of Eq.(6) appears to miss a multiplicative factor of $1 + (2/5)\partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1 \nu)|_{\alpha=1}$. Checking the calculation of [4], we found the missing multiplicative factor in their Eqs. (A50) and (A51), in the form of $q(r)\text{tr}(\tilde{\zeta}_i^2)$. This term was fortuitously omitted in their final expression of $\xi_{\text{pk}}(r)$. Consequently, the correct answer is

$$\begin{aligned} \xi_{\text{pk}}(\nu, R_s, r) = & (\tilde{\zeta}_I^2 \xi_0^{(0)}) + \frac{1}{2}(\xi_0^{(0)} \tilde{\zeta}_{II}^2 \xi_0^{(0)}) - \frac{3}{\sigma_1^2}(\xi_1^{(1/2)} \tilde{\zeta}_{II} \xi_1^{(1/2)}) - \frac{5}{\sigma_2^2}(\xi_2^{(1)} \tilde{\zeta}_{II} \xi_2^{(1)}) \left(1 + \frac{2}{5} \partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1 \nu) \Big|_{\alpha=1}\right) \\ & + \frac{5}{2\sigma_2^4} \left[(\xi_0^{(2)})^2 + \frac{10}{7}(\xi_2^{(2)})^2 + \frac{18}{7}(\xi_4^{(2)})^2 \right] \left(1 + \frac{2}{5} \partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1 \nu) \Big|_{\alpha=1}\right)^2 \\ & + \frac{3}{2\sigma_1^4} \left[(\xi_0^{(1)})^2 + 2(\xi_2^{(1)})^2 \right] + \frac{3}{\sigma_1^2 \sigma_2^2} \left[3(\xi_3^{(3/2)})^2 + 2(\xi_1^{(3/2)})^2 \right] \left(1 + \frac{2}{5} \partial_\alpha \ln G_0^{(\alpha)}(\gamma_1, \gamma_1 \nu) \Big|_{\alpha=1}\right). \end{aligned} \quad (61)$$

We note that this omission has an impact only on the small-scale ($r \lesssim 20 h^{-1}\text{Mpc}$) peak correlation displayed in Fig.2 of their paper. Their results concerning the peak-background split, the gravitational evolution of $\xi_{\text{pk}}(r)$ or the scale-dependence of bias around the Baryon Acoustic Oscillation are unaffected.

Even though we cannot write down a relation of the form $n_{\text{pk}} = \mathcal{F}_X(\delta)$, the peak correlation function up to second-order can nonetheless be thought of as arising from a local bias expansion $\delta_{\text{pk}} = \mathcal{F}_X(\delta_s, \dots)$, i.e.

$$\begin{aligned} \delta_{\text{pk}}(\mathbf{x}) = & b_{10}\delta_s(\mathbf{x}) - b_{01}\nabla^2\delta_s(\mathbf{x}) + \frac{1}{2}b_{20}\delta_s^2(\mathbf{x}) \\ & - b_{11}\delta_s(\mathbf{x})\nabla^2\delta_s(\mathbf{x}) + \frac{1}{2}b_{02}[\nabla^2\delta_s(\mathbf{x})]^2 \\ & + \chi_{10}(\nabla\delta_s)^2(\mathbf{x}) + \frac{1}{2}\chi_{01}[3\partial_i\partial_j\delta_s - \delta_{ij}\nabla^2\delta_s]^2(\mathbf{x}), \end{aligned} \quad (62)$$

provided that we ignore all the contributions involving moments at zero lag. Nonlocality enters through the filtering solely (which is the reason why we still call it a local expansion). As emphasized in [4], it is important to realize that δ_{pk} is *not* a count-in-cell quantity. Counts-in-cells can generally be constructed using the void generating function, see e.g. [28, 29], but it is beyond the scope of this paper to compute moments of the peak frequency distribution function. Since all the bias factors are peak-background split biases obtained from a suitable average of orthogonal polynomials, one could try to write down an expansion in terms of orthogonal polynomials in the variables δ_s , $\nabla^2\delta_s$ etc. such that all the contributions involving moments at zero lag cancel out. We will explore this possibility in future work. Note that this idea was put forward for the first time by [30], who considered correlations of regions above threshold as a

proxy for luminous tracers. More recently, [31] proposed an algorithm based on Hermite polynomials to extract k -dependent bias factors from cross-correlations between the halo and Hermite-transformed mass density field.

To make connection with the formalism of [27] (see also [32]), we define the Fourier space peak bias parameters $c_n(\mathbf{k}_1, \dots, \mathbf{k}_n)$ as the sum over all the contributions to $\delta_{\text{pk}}(\mathbf{x})$ from a given order. We thus have

$$c_1(\mathbf{k}) \equiv (b_{10} + b_{01}k^2) W(kR_s) \quad (63)$$

and

$$c_2(\mathbf{k}_1, \mathbf{k}_2) \equiv \left\{ b_{20} + b_{11}(k_1^2 + k_2^2) + b_{02}k_1^2k_2^2 \right. \\ \left. - 2\chi_{10}(\mathbf{k}_1 \cdot \mathbf{k}_2) + \chi_{01} \left[3(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 - k_1^2k_2^2 \right] \right\} W(k_1R_s)W(k_2R_s), \quad (64)$$

where $W(kR_s)$ is the smoothing kernel. These definitions are consistent with those of the “renormalized” bias parameters introduced by [27] who argued that, owing to rotational symmetry, the peak bias parameters should take the above functional form. In particular, the correspondence between the bias factors associated with η^2 and ζ^2 and those of [27] is $E_2 = 3\chi_{01}$, $C_2 = -2\chi_{10}$ and $D_2 = b_{02} - \chi_{01}$. We stress, however, that the peak bias factors discussed in this work have not been obtained by means of a renormalization procedure. We speculate that the local bias expansion Eq.(62) can be extended to all orders to match the exact peak 2-point correlation function at all separations. Namely,

$$\delta_{\text{pk}}(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3} \dots \frac{d^3k_n}{(2\pi)^n} c_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \\ \times \delta(\mathbf{k}_1) \dots \delta(\mathbf{k}_n) e^{i(\mathbf{k}_1 + \dots + \mathbf{k}_n) \cdot \mathbf{x}}, \quad (65)$$

where $c_n(\mathbf{k}_1, \dots, \mathbf{k}_n)$ is a sum over all the possible combinations of rotational invariants involving exactly n powers of the linear density field δ_s and/or its derivatives.

Our peak-background split approach provides a simple way of predicting the bias coefficients associated with any rotational invariant quantity. The Hermite weighting scheme introduced by [31] furnishes a practical way of measuring the biases b_{ij} from simulations. Clearly, their scheme could be extended to also measure the biases χ_{ij} . However, because discrete density maxima require a somewhat more sophisticated treatment of counts-in-cells, we leave this for future work. Here, we merely establish a recursion relation between the b_{ij} by considering either the property

$$\left(\frac{\partial}{\partial \nu} + \gamma_1 \frac{\partial}{\partial u} \right) e^{-Q_1(\mathbf{y})} = -\nu e^{-Q_1(\mathbf{y})}, \quad (66)$$

or the generating function in Eq.(51). In the latter case, upon substituting $b_u = (\sigma_0/\sigma_1)^2(\nu/\sigma_0 - b_\nu)$ in the expo-

nential factor, we find

$$\delta_c b_{01} = \left(\frac{\sigma_0}{\sigma_1} \right)^2 (\nu^2 - \delta_c b_{10}) \quad (67)$$

at the first order whereas, at the second order, we obtain

$$\delta_c^2 b_{11} = \delta_c^2 \overline{b_\nu b_u} + \frac{\delta_c^2 \gamma_1^2}{(1 - \gamma_1^2)} \\ = \left(\frac{\sigma_0}{\sigma_1} \right)^2 \left[\delta_c^2 \left(\frac{\nu}{\sigma_0} \right) \overline{b_\nu} - \delta_c^2 \overline{b_\nu^2} + \frac{\gamma_1^2 \nu^2}{(1 - \gamma_1^2)} \right] \\ = \left(\frac{\sigma_0}{\sigma_1} \right)^2 \left(-\nu^2 + \nu^2 \delta_c b_{10} - \delta_c^2 b_{20} \right), \quad (68)$$

and

$$\delta_c^2 b_{02} = \delta_c^2 \overline{b_u^2} - \frac{\delta_c^2}{\sigma_2^2(1 - \gamma_1^2)} \\ = \left(\frac{\sigma_0}{\sigma_1} \right)^4 \left[\nu^4 - 2\delta_c^2 \left(\frac{\nu}{\sigma_0} \right) \overline{b_\nu} + \overline{b_\nu^2} - \frac{\gamma_1^2 \nu^2}{(1 - \gamma_1^2)} \right] \\ = \left(\frac{\sigma_0}{\sigma_1} \right)^4 \left[\nu^2 (\nu^2 + 1) - 2\nu^2 \delta_c b_{10} + \delta_c^2 b_{20} \right]. \quad (69)$$

At least for $n = 1, 2$, b_{kl} with $1 \leq l \leq k \leq n$ can be expressed as a linear combination of b_{k0} , $1 \leq k \leq n$, plus a polynomial in ν . This agrees with the findings of [31] obtained within the excursion set approach [31] (except for the multiplicative factors of $(\sigma_0/\sigma_1)^{2l}$). Similar relations should hold at any order in the peak bias parameters b_{ij} . This structure arises from the fact that the peak-background split acts on probability densities, which are continuous functions of space. For discrete tracers such as density maxima, the background perturbation affects the fields appearing in the Gaussian multivariate $P_1(\mathbf{y})$, but not those entering the expression of the peak number density $n_{\text{pk}}(\mathbf{y})$. The peak constraint weights the peak-background split series expansion such that the peak bias factors are recovered.

E. Correlation functions for excursion set peaks

To make connection with the clustering of dark matter halos, we must ensure that the density in a tophat region centered at the peak location never reaches the collapse threshold (δ_c) on any smoothing scale $R > R_s$. Ref. [33] showed that enforcing the conditions $\delta(R_s) > \delta_c$ and $\delta(R_s + \Delta R_s) < \delta_c$ as in [34] provides a very good approximation to the first-crossing distribution when the stochastic walks generated from the variation of R_s are strongly correlated. An important consequence of this result is the possibility of restricting the excursion set to those locations that meet the peak constraint [12]. The number density of dark matter halos per unit mass and volume is usually written

$$\bar{n}(M) = \frac{\bar{\rho}}{M} f(\nu) \frac{d\nu}{dM}, \quad (70)$$

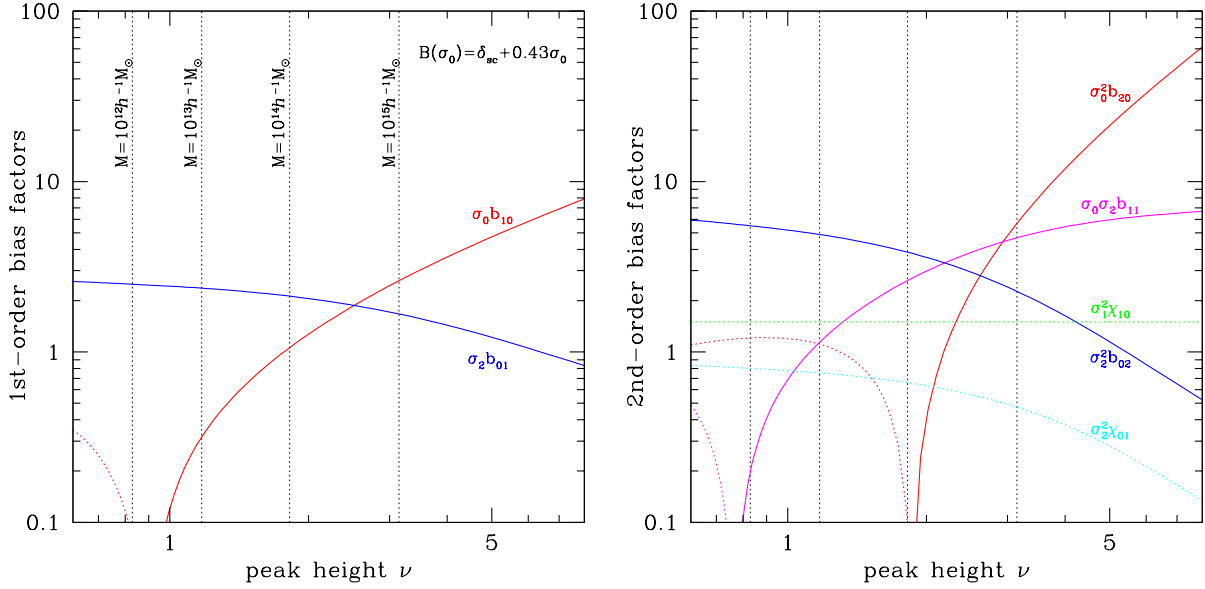


FIG. 1: First-order (left panel) and second-order (right panel) Lagrangian peak bias factors derived from the excursion set peaks mass function Eq.(72) assuming a moving barrier $B(\sigma_0) = \delta_c + 0.43\sigma_0$ (see text). The bias parameters have been multiplied by the appropriate factors of σ_i , so that they all are dimensionless and multiply combinations of normalized variables in the effective local bias expansion Eq.(62). In particular, the second-order bias factors $\sigma_0^2 b_{20}$, $\sigma_0 \sigma_2 b_{11}$ and $\sigma_2^2 b_{02}$ are associated with $\nu(\mathbf{x})$ and $u(\mathbf{x})$, whereas $\sigma_1^2 \chi_{10}$ and $\sigma_2^2 \chi_{01}$ weight the contributions from $\eta^2(\mathbf{x})$ and $\zeta^2(\mathbf{x})$, respectively. Vertical lines mark the peak significance at which the halo mass is $M = 10^{12}, 10^{13}, 10^{14}$ and $10^{15} M_\odot/h$ (from left to right). Dashed curves indicate negative values. Results are shown at $z = 0$ for a Λ CDM cosmology with normalisation $\sigma_8 = 0.82$.

where $f(\nu)$ is the multiplicity function. Following the approach of [12, 34], the number density of peaks identified on the filtering scale R_s and satisfying the aforementioned conditions is

$$\bar{n}_{\text{ESP}}(R_s) dR_s = \frac{3^{3/2}}{R_1^3} \left(\frac{\sigma_2}{\sigma_0} \right) \int d^{10} \mathbf{y} n_{\text{pk}}(\mathbf{y}) u P_1(\mathbf{y}) R_s dR_s. \quad (71)$$

where, for simplicity, we have assumed that the smoothing kernel is Gaussian, but it is straightforward to generalize these results to arbitrary filters. Therefore, $d\nu/dR_s = \nu R_s (\sigma_1/\sigma_0)^2$ and we can write the excursion set peaks multiplicity function as

$$\begin{aligned} f_{\text{ESP}}(\nu) &= \left(\frac{M}{\bar{\rho}} \right) \bar{n}_{\text{ESP}}(R_s) \frac{dR_s}{d\nu} \\ &= \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} \left(\frac{V}{V_*} \right) \frac{G_1^{(1)}(\gamma_1, \gamma_1 \nu)}{\gamma_1 \nu} \\ &= V \bar{n}_{\text{ESP}}(\nu). \end{aligned} \quad (72)$$

Here, $V \equiv M/\bar{\rho}$ is the Lagrangian volume associated with the filter (usually tophat). f_{ESP} is the fundamental ingredient in the mass function prediction of [13] since it can be interpreted as a multiplicity function. It is pretty clear that, in the peak 2-point correlation, the constraint $\delta(R_s) > \delta_c$ and $\delta(R_s + \Delta R_s) < \delta_c$ will translate into an extra multiplicative factor of $(\sigma_2/\sigma_0) u R_s dR_s$ in the integrand. Therefore, the ESP correlation functions can also

be obtained in the exact same way as that of the BBKS peaks, but with a peak number density

$$\begin{aligned} n_{\text{ESP}}(\nu', R_s, \mathbf{x}) &\equiv \frac{3^{3/2}}{R_*^3} \left(-\frac{\text{tr} \zeta}{\gamma_1 \nu'} \right) |\det \zeta| \delta_D(\boldsymbol{\eta}) \\ &\times \theta_H(\lambda_3) \delta_D(\nu - \nu'). \end{aligned} \quad (73)$$

The bias parameters for excursion set peaks are thus given by

$$\begin{aligned} b_{ij} &= \frac{1}{\sigma_0^i \sigma_2^j \bar{n}_{\text{ESP}}} \int d^{10} \mathbf{y} n_{\text{ESP}}(\mathbf{y}) H_{ij}(\nu, u) P_1(\mathbf{y}) \\ \chi_{k0} &= \frac{(-1)^k}{\sigma_1^{2k} \bar{n}_{\text{ESP}}} \int d^{10} \mathbf{y} n_{\text{ESP}}(\mathbf{y}) L_k^{(1/2)} \left(\frac{3\eta^2}{2} \right) P_1(\mathbf{y}) \\ \chi_{0k} &= \frac{(-1)^k}{\sigma_2^{2k} \bar{n}_{\text{ESP}}} \int d^{10} \mathbf{y} n_{\text{ESP}}(\mathbf{y}) L_k^{(3/2)} \left(\frac{5\zeta^2}{2} \right) P_1(\mathbf{y}). \end{aligned} \quad (74)$$

They are similar to the bias factors of BBKS peaks, except for the fact that the k th order moment u^k of the peak curvature must be replaced by $\overline{u^{k+1}}/\bar{u}$ in Eqs.(12) – (14), and $G_0^{(\alpha)}$ must be replaced by $G_1^{(\alpha)}$ in Eq.(49). For large values of $\omega = \gamma_1 \nu$, $G_1^{(\alpha)}$ asymptotes to $G_1^{(\alpha)} \approx \alpha^{-5/2} \omega^4$. This implies that $\partial_\alpha \ln G_1^{(\alpha)}(\gamma_1, \omega)$ converges towards $-5/2$ in the limit $\omega \rightarrow \infty$. Hence, the second-order bias induced by the asymmetry of the peak profile also vanishes in the high peak limit for the ESP multiplicity function, as it should be.

To illustrate the behaviour of the ESP bias parameters, we follow the methodology adopted in [13]. Namely, we smooth the density field with a tophat filter, while sticking to the Gaussian filter to define η_i and ζ_{ij} (so that the second spectral moment σ_2 remains finite). Therefore, we have $\gamma_1 = \sigma_{1\times}^2/(\sigma_{0T}\sigma_{2G})$, where the subscript T and G refer to tophat and Gaussian filtering, respectively, and \times denotes a mixed filtering, i.e. one filter is Gaussian and the other is tophat. Next, we construct the mapping between R_T and R_G by finding the $R_G(R_T)$ for which $\langle\delta_G\delta_T\rangle = \langle\delta_T^2\rangle$. Finally, to account for departures from the spherical collapse approximation, we consider a moving barrier of the (square-root) form $B = \delta_{sc} + \beta\sigma_0$, where $\beta = 0.43$ [35]. For simplicity, we will ignore the scatter around B even though it is quite substantial in the range of ν we are interested in. Fig.1 shows the first- and second-order peak bias factors at $z = 0$ for a Λ CDM cosmology with $\sigma_8 = 0.82$. Note that, while $M = 10^{13} \text{ M}_\odot/h$ translate into a significance of $\nu \approx 1.2$, the actual height of $\nu = 1.2$ density peaks is $B/\sigma_0 \approx 1.6$.

These results can be generalized to arbitrary filtering of the mass density field. For non-Gaussian initial conditions, there are a couple of subtleties which will be discussed elsewhere [45].

IV. DISCUSSION AND CONCLUSIONS

We have shown that the 2-point correlation function $\xi_{\text{pk}}(r)$ of discrete density peaks can be computed, up to second order at least, from an *effective* local bias expansion in continuous fields that are invariant under rotations of the coordinate frame. This local expansion is not a count-in-cell relation in the sense that $\delta_{\text{pk}}(\mathbf{x})$ is merely an effective overdensity that can be used to recover the true $\xi_{\text{pk}}(r)$ from a trivial evaluation of $\langle\delta_{\text{pk}}(\mathbf{x}_1)\delta_{\text{pk}}(\mathbf{x}_2)\rangle$. One of the consequences is that there only is one physically motivated smothing scale: the Lagrangian radius R_s of the halos. Yet another important difference with the widespread local bias model is that one shall ignore all the contributions from zero-lag moments in order to obtain the correct $\xi_{\text{pk}}(r)$.

All the bias coefficients can be derived from a peak-background split argument in which the background perturbation shifts the zero mean of the 1-point probability distribution functions of the rotationally invariant fields, unlike essentially all the other peak-background split formulations which consider a change in the number density of the tracers. Consequently, it is possible to derive bias factors from a peak-background split argument even if the variables are integrated over. The resulting probability densities can then be expanded in orthogonal polynomial bases. For the normally distributed peak height $\nu(\mathbf{x})$ and curvature $u(\mathbf{x})$, these are bivariate Hermite polynomials whereas, for the chi-squared distributed $\eta^2(\mathbf{x})$ and $\zeta^2(\mathbf{x})$, these are generalized Laguerre polynomials.

The peak bias factors are then obtained upon averaging the appropriate orthogonal polynomials over all the spatial locations that satisfy the peak constraint.

We have demonstrated that our simple local expansion reproduces the 2-point peak correlation function $\xi_{\text{pk}}(r)$ computed at second order by [4] after a tedious expansion of the joint probability density $P_2(\mathbf{y}_1, \mathbf{y}_2; r)$. We believe that it should remain valid at higher orders. Furthermore, because discreteness enters the calculation only when averaging the orthogonal polynomials, we speculate that this local bias expansion combined with the peak-background split approach presented here can be generalized to describe the clustering of any point process of a Gaussian random field. The great advantage of our approach is that it circumvents the computation of $P_2(\mathbf{y}_1, \mathbf{y}_2; r)$, and requires only the evaluation of $P_1(\mathbf{y})$.

Our approach can be easily generalized to more sophisticated constraints involving, for instance, the tidal shear $\partial_i\partial_j\Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is the gravitational potential. As noted in [19, 22], the quadratic invariant $s^2(\mathbf{x})$ (the equivalent of our $\zeta^2(\mathbf{x})$ but with δ replaced by Φ) follows a χ^2 -distribution with 5 degrees of freedom. Therefore, we expect that its associated bias parameters are given by some suitable average of the Laguerre polynomials $L_k^{(3/2)}(x)$. If the (nonspherical) collapse occurs at the spatial location of density peaks or includes the dependence on the large scale environment, then the χ_5^2 distribution will be replaced by the appropriate conditional probability density [36, 37], to which we shall apply the peak-background split in order to read off the new bias parameters.

Corrections induced by nonlinear gravitational evolution can also be decomposed into rotational invariants [38–40]. Therefore, if one ignores the diffusion kernels (i.e. the propagators introduced by [41]), then it is straightforward to find explicit expressions for the Eulerian bias parameters in terms of local and nonlocal Lagrangian bias factors [19, 42–44]. This procedure can clearly be applied to our effective bias expansion Eq.(62), with the important caveat that discrete density peaks exhibit a statistical velocity bias. [9]. Notwithstanding this, we expect from the structure of the F_2 kernel that the bias factors χ_{ij} remain constant with time, in agreement with the findings of [4]. For a more realistic treatment of gravitational motions, it should be possible to compute the evolved 2-point peak correlation $\xi_{\text{pk}}(r, z)$ in the framework of the integrated perturbation theory proposed by [27]. We leave all this to future work.

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Appendix A: Computing correlators

For illustration, we evaluate the cross-covariance $\langle \eta_1^2 \eta_2^2 \rangle$ of the square modulus of the gradient $\eta^2(\mathbf{x})$ at two different spatial locations \mathbf{x}_1 and \mathbf{x}_2 . We have

$$\begin{aligned}
 \langle \eta_1^2 \eta_2^2 \rangle &= \frac{1}{\sigma_1^4} \left\{ \prod_{i=1}^4 \int \frac{d^3 k_i}{(2\pi)^3} \right\} (\mathbf{k}_1 \cdot \mathbf{k}_2) (\mathbf{k}_3 \cdot \mathbf{k}_4) \quad (\text{A1}) \\
 &\quad \times \langle \delta_s(\mathbf{k}_1) \delta_s(\mathbf{k}_2) \delta_s(\mathbf{k}_3) \delta_s(\mathbf{k}_4) \rangle e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}_2 + i(\mathbf{k}_3 + \mathbf{k}_4) \cdot \mathbf{x}_1} \\
 &= 1 + \frac{2}{\sigma_1^4} \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 P_s(k_1) P_s(k_2) \\
 &\quad \times e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} \\
 &= 1 + \left(\frac{2}{\sigma_1^4} \right) \delta_{ij} \delta_{lm} J_{il}(r) J_{jm}(r) ,
 \end{aligned}$$

where

$$J_{ij}(r) = \int \frac{d^3 k}{(2\pi)^3} k_i k_j P_s(k) e^{i\mathbf{k} \cdot \mathbf{r}} . \quad (\text{A2})$$

To evaluate $J_{ij}(r)$, we express k_i in terms of the components $\hat{k}_i = k_i/k$ of the unit vector, and take advantage of the fact that the integral over the angular variables is

$$\begin{aligned}
 &\frac{1}{4\pi} \int d\Omega_{\hat{\mathbf{k}}} \hat{k}_i \hat{k}_j e^{i\mathbf{k} \cdot \mathbf{r}} \\
 &= \frac{1}{3} \left[j_0(kr) + j_2(kr) \right] \delta_{ij} - j_2(kr) \hat{r}_i \hat{r}_j . \quad (\text{A3})
 \end{aligned}$$

Therefore, the product $\delta_{ij} \delta_{lm} J_{il}(r) J_{jm}(r)$ becomes (we omit the r -dependence for conciseness)

$$\begin{aligned}
 \delta_{ij} \delta_{lm} J_{ij} J_{lm} &= \delta_{ij} \delta_{lm} \left[\frac{1}{3} (\xi_0^{(1)} + \xi_2^{(1)}) \delta_{il} - \xi_2^{(1)} \hat{r}_i \hat{r}_l \right] \\
 &\quad \times \left[\frac{1}{3} (\xi_0^{(1)} + \xi_2^{(1)}) \delta_{jm} - \xi_2^{(1)} \hat{r}_j \hat{r}_m \right] \\
 &= \frac{1}{3} (\xi_0^{(1)} + \xi_2^{(1)})^2 - \frac{2}{3} \xi_2^{(1)} (\xi_0^{(1)} + \xi_2^{(1)}) + (\xi_2^{(1)})^2 \\
 &= \frac{1}{3} \left[(\xi_0^{(1)})^2 + 2(\xi_2^{(1)})^2 \right] , \quad (\text{A4})
 \end{aligned}$$

which yields Eq.(29) once the multiplicative factor of $(2/\sigma_1^4)$ and the additive zero-lag contribution are accounted for.

Appendix B: Shape factor for peaks

In terms of the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the hessian matrix ζ_{ij} , the asymmetry parameters that quantify the departure from a spherically symmetric peak density profile are $v = (\lambda_1 - \lambda_3)/2$ and $w = (\lambda_1 - 2\lambda_2 + \lambda_3)/2$. The peak constraint together, with our choice of ordering, impose the four conditions $v \geq 0$, $-v \leq w \leq v$, $(u + w) \geq 3v$ and $u \geq 0$. Following [3], we also introduce an auxiliary function that measures the degree

of asphericity expected for a peak,

$$F(u, v, w) = (u - 2w) \left[(u + w)^2 - 9v^2 \right] v (v^2 - w^2) \quad (\text{B1})$$

This function scales as $\propto u^3$ in the limit $u \gg 1$.

In [4], the peak 2-point correlation up to second order (i.e. terms quadratic in the correlation of the density field and its derivatives) is written as the sum of the linear contribution and three second-order terms $\xi_{\text{pk}}^{(2i)}$, $i = 1, 2, 3$. In particular, $\xi_{\text{pk}}^{(22)}$ contains all the terms for which the ν -dependence cannot be expressed as a polynomial in the linear and quadratic bias parameters b_{ij} . Their expression is phrased in terms of

$$\begin{aligned} f(u, \alpha) &\equiv \frac{3^2 5^{5/2}}{\sqrt{2\pi}} \left\{ \int_0^{u/4} dv \int_{-v}^{+v} dw + \int_{u/4}^{u/2} dv \int_{3v-w}^v dw \right\} F(u, v, w) e^{-\frac{5\alpha}{2}(3v^2+w^2)} \\ &= \frac{1}{\alpha^4} \left\{ \frac{e^{-5\alpha u^2/2}}{\sqrt{10\pi}} \left(-\frac{16}{5} + \alpha u^2 \right) + \frac{e^{-5\alpha u^2/8}}{\sqrt{10\pi}} \left(\frac{16}{5} + \frac{31}{2} \alpha u^2 \right) \right. \\ &\quad \left. + \frac{\sqrt{\alpha}}{2} (\alpha u^3 - 3u) \left[\text{Erf} \left(\sqrt{\frac{5\alpha}{2}} \frac{u}{2} \right) + \text{Erf} \left(\sqrt{\frac{5\alpha}{2}} u \right) \right] \right\}, \quad (\text{B2}) \end{aligned}$$

and its integral over the n th power of the peak curvature u times the u -dependent part of the one-point probability distribution,

$$G_n^{(\alpha)}(\gamma_1, w) = \int_0^\infty dx x^n f(x, \alpha) \frac{e^{-(x-w)^2/2(1-\gamma_1^2)}}{\sqrt{2\pi(1-\gamma_1^2)}}. \quad (\text{B3})$$

These functions are very similar, albeit more general than those defined in Eqs (A15) and (A19) of [3].

Appendix C: Non-central chi-squared distributions

The probability density of a non-central chi-squared distribution $\chi_k'^2(x; \lambda)$ with k degrees of freedom and non-centrality parameter λ is given by

$$\chi_k'^2(x; \lambda) = \frac{e^{-(x+\lambda)/2}}{2} \left(\frac{x}{\lambda} \right)^{\alpha/2} I_\alpha \left(\sqrt{\lambda x} \right), \quad (\text{C1})$$

where $\alpha = k/2 - 1$ and $I_\alpha(x)$ is a modified Bessel function of the first kind. Ref. [26] proposed the following Laguerre polynomial expansion,

$$\begin{aligned} \chi_k'^2(x; \lambda) &= \frac{e^{-x/2}}{2} \left(\frac{x}{2} \right)^\alpha \sum_{j=0}^\infty \frac{(-\frac{\lambda}{2})^j}{\Gamma(\frac{1}{2}k + j)} L_j^{(\alpha)} \left(\frac{x}{2} \right) \quad (\text{C2}) \\ &= \chi_k^2(x) \sum_{j=0}^\infty \frac{\Gamma(\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu + j)} \left(-\frac{\lambda}{2} \right)^j L_j^{(\alpha)} \left(\frac{x}{2} \right). \end{aligned}$$

The non-central χ^2 -distribution can also be represented as a Poisson-weighted mixture of central χ^2 -distributions (this was used by, e.g., [19] to estimate the nonlocal Lagrangian bias induced by ellipsoidal collapse). Note, however, that this representation does not make apparent the connection with the bias parameters χ_{ij} .